Spectral parameter-dependent approach to quantized algebra, its multiparameter deformations and their q-oscillator realizations

1992 J. Phys. A: Math. Gen. 25 4147

(http://iopscience.iop.org/0305-4470/25/15/022)

J. Phys. A: Math. Gen. 25 (1992) 4147–4156. Printed in the UK

Spectral parameter-dependent approach to quantized algebra, its multiparameter deformations and their q-oscillator realizations

B Basu Mallick and A Kundu

Theoretical Nuclear Physics Division, Saha Institute of Nuclear Physics, Block AF, Sector 1, Bidhannagar Calcutta-700 064, India

Received 16 December 1991

Abstract. Yang–Baxterization of Faddeev–Reshetikhin–Takhtajan (FRT) algebra leading to quantum YBE is considered for a class of R-matrices which satisfy Hecke relations. We apply such construction to the $gl(N)$ case together with its multiparameter deformations revealing the connection between FRT relations and the extended trigonometric Sklyanin algebra. New realizations of FRT algebra through q-oscillator modes are presented, and their potential importance in the context of quantum-integrable models is discussed. A multiparametrized R-matrix with the inclusion of spectral parameter is constructed as a by-product.

1. Introduction

The theory of quantized algebra and the related quantum group structures have now become a well established discipline in mathematical physics with a wide range of applications [1]. These structures have appeared in the literature in several related forms [2–5]. A rigorous mathematical formulation was given by Jimbo [4] and Drinfeld [5] by defining them as one-parameter deformations of a universal enveloping algebra of a Lie algebra. This rather formal approach was later put into a more physical form by Faddeev, Reshetikhin and Takhtajan (FRT) [6], which was closer to the theory of integrable systems and based on the notion of quantization of the function space defined on the quantized manifold of a Lie group. Exploiting the duality condition of a Hopf algebra, the FRT algebra may be formulated through some basic relations like

$$R^+ L_1^{(\pm)} L_2^{(\pm)} = L_2^{(\pm)} L_1^{(\pm)} R^+ \tag{1.1a}$$

$$R^+ L_1^{(+)} L_2^{(-)} = L_2^{(-)} L_1^{(+)} R^+ \, . \tag{1.1b}$$

The associativity condition of the above algebra gives

$$R^+_{12} R^+_{13} R^+_{23} = R^+_{23} R^+_{13} R^+_{12} \tag{1.2}$$

leading to the braid group representation for the matrix $\hat{R}^+ = \mathcal{P} R^+$, $\mathcal{P}$ being the permutation operator. These algebraic relations are quite general and able to produce quantum algebras related to various Lie groups as different realizations, depending on the corresponding solutions of the R^{+}-matrices [7, 8].

However some natural questions may arise at this juncture. Firstly, we note that the notion of a spectral parameter is absent in the above picture and therefore we may ask whether a spectral parameter-dependent approach can be adopted for extracting all these seemingly different-looking relations from a single equation. Another interesting question is why only the relation (1.1*b*) appears in the FRT algebra and not its complementary one, $R^{+}L_1^{(-)}L_2^{(+)} = L_2^{(+)}L_1^{(-)}R^{+}$. This latter point was already raised by Macfarlane and explained by Burroughs [8] as being due to the consequence of quasitriangularity of the quantum double.

In the subsequent development of this subject, a consistent multiparameter deformation of the R-matrix and the corresponding quantum group were suggested by several authors [9–11]. Since such deformations of R-matrix and $L^{(\pm)}$ operators do not affect the algebraic structure (1.1*a*) and (1.1*b*), the questions raised above with emphasis on the parametrization problem are also relevant for this deformed case.

For addressing these problems we focus on the standard quantum Yang–Baxter equation (QYBE) involving spectral parameters λ, μ:

$$R(\lambda-\mu)L_1(\lambda)L_2(\mu) = L_2(\mu)L_1(\lambda)R(\lambda-\mu) \qquad L_1 \equiv L\otimes 1 \quad L_2 \equiv 1\otimes L \tag{1.3}$$

and explore its relation with the FRT algebra. The $R(\lambda)$-matrix appearing in (1.3) satisfies a scalar-type Yang–Baxter equation (YBE)

$$R_{12}(\lambda)R_{13}(\lambda+\mu)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda+\mu)R_{12}(\lambda)\,. \tag{1.4}$$

The above QYBE is a key relation in quantum-integrable theory, where $L(\lambda)$ having non-commuting matrix elements represents the related Lax operator. The observation that YBE (1.4) is similar to the relation (1.2) and goes to it in the limit $\lambda, \mu \to \infty$, might serve as a motivation for also obtaining the FRT relations at some limits of the spectral parameters from QYBE (1.3). A generalized $L(\lambda)$ operator was recently considered by us [12] for generating a class of quantum-integrable models through an extension of the Sklyanin algebra [3] at its trigonometric limit. Curiously we find that the same $L(\lambda)$ operator, corresponding to the $R(\lambda)$-matrix in the fundamental representation of $gl(N)$, is able to reproduce the FRT relations from QYBE, along with an additional equation related to the form of Macfarlane's query.

Looking now to the same problem in the reverse way opens up, interestingly, a possibility of Yang–Baxterization of the FRT algebra leading to the QYBE (1.3). It appears that the Yang–Baxterization related to QYBE, an important construction in the context of quantum integrable models, has received much less attention compared to the usual Yang–Baxterization [13, 14] leading to the YBE (1.4). We observe that for the specfic forms of $R(\lambda)$ and $L(\lambda)$ considered here, such Yang–Baxterization may be carried through only when the corresponding $\hat{R}^{\pm}$-matrices satisfy the Hecke relations. It also turns out that this extra Hecke relation is equivalent to the 'initial condition' of the $R(\lambda)$-matrix and relates the FRT algebra with the additional equation obtained by us. Coming now to the multiparameter deformed cases, one may note that such deformations of R-matrices for the fundamental representation of $gl(N)$ equally satisfy the Hecke algebra and therefore the above Yang–Baxterization scheme of FRT algebra is also applicable to these deformed cases leading to the QYBE. This naturally shows that the multiparameter deformations, suggested basically for the spectral parameterless R-matrices, can also be extended to include such parameters.

This Yang–Baxterization reveals that the extended trigonometric Sklyanin algebra, obtained earlier in connection with the integrable models [12], is intimately related to the FRT algebra. We show further that such Sklyanin algebra generalized for $gl(N)$ and its multiparameter deformations can be realized through q-oscillator modes introduced by several authors in the recent past [15], which also consequently gives new realizations of the FRT algebra in q-oscillators. The contents of the paper is as indicated in the titles of the subsequent four sections.

2. Relation of FRT algebra with QYBE and extended trigonometric Sklyanin algebra

Since our objective here is to explore the relation of the FRT algebra (1.1) with QYBE (1.3) involving spectral parameters, we intend to follow somewhat similar reasoning of Sklyanin, where a form of $L(\lambda)$ operator was suggested for deriving a spectral parameterless quadratic algebra [3]. In particular, for the known R-matrix corresponding to the spin-$\frac{1}{2}$ XXZ chain

$$R(\lambda)=\sum_{i=0}^{3} w_i\sigma_i\otimes\sigma_i \qquad \sigma_0\equiv\mathbf{1}$$
$$w_1=w_2=\frac{1}{2} \qquad w_0+w_3=\frac{\sin\alpha(\lambda+1)}{\sin\alpha} \qquad w_0-w_3=\frac{\sin\alpha\lambda}{\sin\alpha} \tag{2.1}$$

the L-operator taken in the form $L=\sum_{i=0}^{3} w_i S_i\sigma_i$, yields from QYBE (1.3) the trigonometric Sklyanin algebra (TSA):

$$[S_0,S_3]=0 \qquad [S_0,S_\pm]=\mp\tan^2(\alpha/2)\,[S_\pm,S_3]_+$$
$$[S_3,S_\pm]=\pm[S_0,S_\pm]_+ \qquad [S_+,S_-]=4S_0S_3\,. \tag{2.2}$$

However we observe that for the same trigonometric R-matrix (2.1) if the $L(\lambda)$-operator solution is given in the form

$$L(\lambda)=\begin{pmatrix}\frac{1}{\xi}\tau_1^+ +\xi\tau_1^- & \tau_{21}\\ \tau_{12} & \frac{1}{\xi}\tau_2^+ +\xi\tau_2^-\end{pmatrix} \tag{2.3}$$

where $\xi=\mathrm{e}^{-\mathrm{i}\alpha\lambda}$, from QYBE one gets another spectral parameter-free algebra

$$[\tau_{12},\tau_{21}]=-2\mathrm{i}\sin\alpha\left(\tau_1^+\tau_2^- -\tau_1^-\tau_2^+\right)$$
$$\tau_i^\pm\tau_{ij}=\mathrm{e}^{\pm\mathrm{i}\alpha}\tau_{ij}\tau_i^\pm \tag{2.4}$$
$$\tau_i^\pm\tau_{ji}=\mathrm{e}^{\mp\mathrm{i}\alpha}\tau_{ji}\tau_i^\pm$$

with all operators $\tau_i^\pm$ commuting among themselves. It is worth noting that the above algebra (2.4) is an extension of the TSA and reduces to it in the particular symmetric case $\tau_2^-=-\tau_1^+$, $\tau_2^+=-\tau_1^-$ with the mapping

$$\tau_1^+ +\tau_1^- =\frac{1}{\cos(\alpha/2)}S_3 \qquad \tau_1^+ -\tau_1^- =-\frac{\mathrm{i}}{\sin(\alpha/2)}S_0 \qquad \tau_{12}=S_+,\tau_{21}=S_-\,.$$

The construction for $L(\lambda)$ (2.3) may be viewed as the insertion of independent operators $\{\tau\}$ in the basis of loop algebra $e_{ij}(n) \equiv \xi^n e_{ij}, i, j \leqslant 2$, with $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$, where terms up to $e(\pm 1)$ only are kept guided by the powers of ξ appearing in the structure of $R(\lambda)$. Note that $L(\lambda)$ in this form may be considered naturally as the Lax operator of some generalized quantum-integrable model corresponding to the R-matrix (2.1). We have found in earlier occasions that different reductions of this particular ancestor model with suitable bosonic (or q-bosonic) realizations are able to generate a series of quantum-integrable systems including the sine–Gordon model, the Liouville model, the massive Thirring model, a novel derivative nonlinear Schrödinger model etc [12].

Interestingly we find that such an $L(\lambda)$-operator may also play significant role in the present context. For this purpose we recall first that under the λ-dependent 'symmetry breaking' transformation [16] $R_{kl}^{mn} \longrightarrow \tilde{R}_{kl}^{mn} = \mathrm{e}^{\mathrm{i}\alpha\lambda(k+n-l-m)/2} R_{kl}^{mn}$ for the charge-conserving case $m + n = k + l$, a new R-matrix is generated from the original one satisfying YBE. It may be noted that such a transformation when expressed in the matrix form looks like a 'gauge transformation', and in particular for the case (2.1) takes the form

$$\tilde{R}(\lambda - \mu) = A(\lambda) \otimes A(\mu) R(\lambda - \mu) A^{-1}(\lambda) \otimes A^{-1}(\mu) \qquad A(\lambda) = \mathrm{e}^{\mathrm{i}\alpha\lambda\sigma_3/2}\,. \tag{2.5}$$

One finds that the $R^{\pm}$-matrices involved in the FRT construction may be obtained from this transformed $\tilde{R}$-matrix in the limit $\lambda \to \pm\mathrm{i}\infty$ and may be expressed as [17]

$$\tilde{R}(\lambda) = \frac{1}{\xi} R^+ - \xi R^- \qquad R^+ = \begin{pmatrix} q & & & \\ & 1 & q - q^{-1} & \\ & 0 & 1 & \\ & & & q \end{pmatrix} \qquad R^- = \mathcal{P}(R^+)^{-1}\mathcal{P}\,. \tag{2.6}$$

where $q = \mathrm{e}^{\mathrm{i}\alpha}$. We observe now that, to be consistent with such transformation, the $L(\lambda)$-operator as a solution of QYBE must also change through a similar 'gauge transformation' $\tilde{L}(\lambda) = A(\lambda)L(\lambda)A^{-1}(\lambda)$, for which our Lax operator (2.3) takes the form

$$\tilde{L}(\lambda) = \frac{1}{\xi} L^{(+)} + \xi L^{(-)} \qquad L^{(+)} = \begin{pmatrix} \tau_1^+ & \tau_{21} \\ 0 & \tau_2^+ \end{pmatrix} \qquad L^{(-)} = \begin{pmatrix} \tau_1^- & 0 \\ \tau_{12} & \tau_2^- \end{pmatrix}\,. \tag{2.7}$$

It is worth observing that the $\tilde{L}(\lambda)$ with operator-valued matrix elements constructed here allows a similar expansion as $\tilde{R}(\lambda)$ in the spectral parameter, where the triangular matrices $L^{(\pm)}$ are analogous to $R^{\pm}$. To make their relation with FRT algebra more transparent we insert the expansion (2.6) and (2.7) into the QYBE (1.3) and match the coefficients in the different powers of the spectral parameters, which finally results the following equations:

$$R^{\pm} L_1^{(\pm)} L_2^{(\pm)} = L_2^{(\pm)} L_1^{(\pm)} R^{\pm} \tag{2.8a}$$

$$R^{\pm} L_1^{(\pm)} L_2^{(\mp)} = L_2^{(\mp)} L_1^{(\pm)} R^{\pm} \tag{2.8b}$$

along with

$$R^+ L_1^{(-)} L_2^{(+)} - L_2^{(+)} L_1^{(-)} R^+ = R^- L_1^{(+)} L_2^{(-)} - L_2^{(-)} L_1^{(+)} R^-\,. \tag{2.9}$$

It is interesting to note that (2.8*a*) and (2.8*b*) are the same equations as the FRT algebra (1.1), while (2.9) is an extra relation. This particular relation demonstrates that $R^+ L_1^{(-)} L_2^{(+)} - L_2^{(+)} L_1^{(-)} R^+ = 0$ does not hold in our case, justifying Macfarlane's query and moreover relates it to another similar expression. Along with the above connection between the FRT algebra and QYBE it is also evident through this construction that the elementwise form of the FRT algebra, expressed through $L^{(\pm)}$ (2.7), represents the same extended trigonometric Sklyanin algebra (2.4).

3. Yang–Baxterization of FRT algebra and its realization through q-oscillators

We try to explore here to what extent the above procedure of relating FRT algebra with a spectral parameter-dependent scheme may be generalized and under which conditions it is possible. Observe that if the same problem is looked at from a different angle, it would be equivalent to the Yang–Baxterization of the FRT algebra. The notion of Yang–Baxterization, i.e. constructing a spectral parameter-dependent $R(\lambda)$-matrix, with commuting matrix elements and satisfying YBE (1.4), was introduced by Jones [13] and subsequently studied by others [14]. On the other hand the Yang–Baxterization leading to QYBE (1.3) involving $L(\lambda)$ with operator-valued matrix elements has not received much attention, though it has immense importance in integrable theories with $L(\lambda)$ playing the role of Lax operators. Our aim is to investigate such possibilities of Yang–Baxterization leading to the QYBE starting from the FRT algebra (1.1) in analogy with the usual approach, which starts from the braid group relations. Since for this purpose one needs to construct $R(\lambda)$ as well as $L(\lambda)$ from objects involved in FRT, we may propose the same forms (2.6), (2.7) to be valid in the general case:

$$\tilde{R}(\lambda) = \frac{1}{\xi} R^+ - \xi R^- \tag{3.1a}$$

$$\tilde{L}(\lambda) = \frac{1}{\xi} L^{(+)} + \xi L^{(-)} \tag{3.1b}$$

where $\xi = \mathrm{e}^{i\kappa\lambda}$, with κ being an arbitrary scaling constant. However we find that for this construction, the use of FRT algebra (1.1*a*) and (1.1*b*) alone is not sufficient for the validity of QYBE (1.3), the inclusion of additional relation (2.9) is also necessary. Naturally one may wonder about the significance of this extra relation and the way it is related to the FRT algebra. We observe that, if on the $R(\lambda)$-matrix construction (3.1*a*) some additional constraint is demanded given by the 'initial' condition $R(0) = c\mathcal{P}$ (relevant to most of the physically important quantum-integrable models) it yields a relation like

$$R^+ - R^- = c\mathcal{P} \,. \tag{3.2}$$

Remarkably, we find that the same spectral-parameterless condition with the use of FRT algebra is able to produce the additional equation (2.9). Therefore the 'initial' condition (3.2) may be taken instead of (2.9) along with FRT algebra for reproducing QYBE through the construction (3.1*a*, *b*). It may also be noted that using the relation $\hat{R}^{\pm} = \mathcal{P} R^{\pm}$ and $\hat{R}^- = (\hat{R}^+)^{-1}$, one can relate (3.2) in turn to $(\hat{R}^+)^2 = c\hat{R}^+ + 1$, which is the crucial extra Hecke relation. Thus we may conclude finally that for the Yang–Baxterization of FRT algebra leading to QYBE presented here, the $\hat{R}^{\pm}$-matrices must also satisfy the Hecke algebra. Similar observation regarding the role of Hecke algebra in the construction of $R(\lambda)$, satisfying YBE, was made earlier by Jones [13].

We would like to show now that the Yang–Baxterization of FRT algebra can be carried through in the general case of $gl(N)$ in their fundamental representations, where $R^{\pm}$-matrices are given by [8]

$$R^+ = q\sum_k e_{kk}\otimes e_{kk} + \sum_{k\neq l} e_{kk}\otimes e_{ll} + (q-q^{-1})\sum_{k<l} e_{kl}\otimes e_{lk}$$
$$R^- = \mathcal{P}(R^+)^{-1}\mathcal{P} \tag{3.3}$$

with $(e_{kl})_{mn} = \delta_{km}\delta_{ln}$ and all the indices running from 1 to N. It is easy to check that the above R^+-matrix satisfies the Hecke algebra and therefore following our above argument the construction (3.1a,b) for $R(\lambda)$ and $L(\lambda)$ should be allowed, where $L^{(\pm)}$ is the upper (lower) triangular matrix form with some as yet unspecified operators $\tau_k^{\pm}$ and τ_{kl} as

$$L^{(+)} = \sum_k \tau_k^+ e_{kk} + \sum_{k<l} \tau_{lk} e_{kl} \qquad L^{(-)} = \sum_k \tau_k^- e_{kk} + \sum_{k>l} \tau_{lk} e_{kl}. \tag{3.4}$$

It is worth noting that the $R(\lambda)$ and $L(\lambda)$ thus constructed, after a λ-dependent 'gauge transformation', may be cast in the same form of [18], where they were used specifically in the context of Toda field theory.

However we see here that any realization of FRT algebra in some physical variables through the above Yang–Baxterization scheme is able to generate a corresponding quantum-integrable model. For finding such realizations of FRT algebra we express it first in the elementwise form using (3.3) and (3.4). The resulting set of relations represent a quadratic algebra and may be grouped in the following way for all different k, l, m, n indices:

$$\tau_k^{\pm}\tau_{kl} = e^{\pm i\alpha}\tau_{kl}\tau_k^{\pm} \tag{3.5a}$$
$$\tau_k^{\pm}\tau_{lk} = e^{\mp i\alpha}\tau_{lk}\tau_k^{\pm} \tag{3.5b}$$
$$[\tau_{kl}, \tau_{lk}] = 2i\sin\alpha(\tau_k^-\tau_l^+ - \tau_k^+\tau_l^-) \tag{3.5c}$$

with all diagonally placed operators $\tau_k^{\pm}$ commuting among themselves in addition to

$$[\tau_k^{\pm}, \tau_{lm}] = 0 \tag{3.6a}$$
$$\tau_{kl}\tau_{km} = e^{i\epsilon\alpha}\tau_{km}\tau_{kl} \tag{3.6b}$$
$$\tau_{kl}\tau_{ml} = e^{-i\epsilon\alpha}\tau_{ml}\tau_{kl} \tag{3.6c}$$
$$[\tau_{mk}, \tau_{kl}] = 2i\epsilon\sin\alpha\tau_k^{(\epsilon)}\tau_{ml} \tag{3.6d}$$

where $\epsilon = \text{sign}(k-l) + \text{sign}(l-m) + \text{sign}(m-k)$, $\tau_k^{(\epsilon)} = \tau_k^{\pm}$ for $\epsilon = \pm 1$ and

$$[\tau_{kl}, \tau_{mn}] = 2i\rho\sin\alpha\tau_{ml}\tau_{kn} \tag{3.7}$$

with $\rho = +1$ for $l > n > k > m$ (and all its cyclic inequalities), while $\rho = -1$ for the reverse inequalities and $\rho = 0$ otherwise. It may be noted that the above relations are the generalization of the extended TSA (2.4) and reduce to it at $N = 2$, when only the relations (3.5) are relevant. The Casimir operators of this algebra may be evaluated from the quantum determinant of $L(\lambda)$ following [3, 19]. Note that a realization of the above algebra in the generators of quantum group $U_q(sl(N))$ may be obtained using the construction of Burroughs [8], while a different realization through canonical bosonic operators may be given from the expressions of the Lax operator of the Toda field model [18]. However we like to present here another interesting realization of the extended TSA (3.5)–(3.7) through recently introduced q-oscillators [15] satisfying the commutation relations

$$[A, n] = A \qquad [A^\dagger, n] = -A^\dagger \qquad AA^\dagger - q^{-1}A^\dagger A = q^n . \tag{3.8}$$

In terms of N independent, mutually commuting q-oscillator modes A_k, this realization looks like

$$\tau_k^\pm = \pm\frac{\mathrm{i}}{2}\mathrm{e}^{\mp\mathrm{i}\alpha n_k} \tag{3.9a}$$

$$\tau_{kl} = \mathrm{e}^{\mathrm{i}\alpha/2}\sin\alpha g_{kl}^{-1} A_l^\dagger A_k \tag{3.9b}$$

$$\tau_{lk} = \mathrm{e}^{-\mathrm{i}\alpha/2}\sin\alpha g_{kl} A_k^\dagger A_l \tag{3.9c}$$

where $k > l$ with $g_{kl} = \exp[(\mathrm{i}\alpha/2)(n_k + n_l + 2\sum_{j=l+1}^{k-1} n_j)]$. One may observe that these expressions are similar to the Schwinger-type realization of quantum group [15]. On the other hand in analogy with the q-Holstein–Primakoff transformation of $U_q(SU(N))$ [20] yet another realization of the extended TSA is possible to construct through only $(N - 1)$ independent q-oscillators. Leaving out the details we just mention that such constructions may be obtained straight away from (3.9) by replacing formally

$$n_1 \to s - \sum_{k=2}^{N} n_k \qquad A_1 \to -\mathrm{e}^{(\mathrm{i}\alpha/2)(1-s)}([n_1]_q)^{\frac{1}{2}}$$

etc.

4. Multiparameter deformation of the extended TSA and its q-oscillator realization

Let us see now whether the above Yang–Baxterization scheme is also applicable to the case of multiparameter quantum groups and the corresponding deformed R-matrices [9–11]. Such deformed spectral-parameterless $R^\pm$-matrices may be considered as 'gauge transformation' of the standard one-parameter version (3.3), which in fundamental representation of $gl(N)$ takes the form [10, 11]

$$\begin{aligned} R^+_{(\phi)} &= q\sum_k e_{kk}\otimes e_{kk} + \sum_{k\neq l}\mathrm{e}^{\mathrm{i}\phi_{lk}} e_{kk}\otimes e_{ll} + (q - q^{-1})\sum_{k<l} e_{kl}\otimes e_{lk} \\ R^-_{(\phi)} &= \mathcal{P}(R^+_\phi)^{-1}\mathcal{P} \end{aligned} \tag{4.1}$$

having $N(N-1)/2$ deforming parameters ϕ_{kl} with the antisymmetric property $\phi_{kl} = -\phi_{lk}$. As explained in section 3, we recall that the extra Hecke relation

(3.2) along with the FRT algebra is necessary for our Yang–Baxterization leading to QYBE. One easily checks that the $R^{\pm}_{(\phi)}$-matrices (4.1) satisfy such Hecke algebra and therefore for this deformed case also we may carry through the construction of λ-dependent $R(\lambda)$ and $L(\lambda)$ using (3.1*a*) and (3.1*b*). The resultant explicit form of $R(\lambda)$ is

$$R_{(\phi)}(\lambda) = (\xi^{-1}q - \xi q^{-1})\sum_{k} e_{kk}\otimes e_{kk} + (\xi^{-1}-\xi)\sum_{k\neq l} e^{i\phi_{lk}} e_{kk}\otimes e_{ll} + (q-q^{-1})\Big(\xi^{-1}\sum_{k<l} e_{kl}\otimes e_{lk} + \xi\sum_{k>l} e_{kl}\otimes e_{lk}\Big) \tag{4.2}$$

which shows, interestingly, that the multiparameter deformation of $R^{\pm}$-matrix [10, 11] can also be extended to the spectral parameter-dependent case. If the constituent $L^{(\pm)}$ operators of the corresponding $L(\lambda)$ are taken in the same form as in the undeformed case (3.4) replacing only τ-operators by $\tilde{\tau}$, the algebraic relations (3.5)–(3.7) suffer significant deformations dictated by the QYBE (1.3), leading to a multiparameter generalization of the extended TSA. The previous Weyl-type relations now acquire extra phase factors in the form

$$\tilde{\tau}^{\pm}_{k}\tilde{\tau}_{kl} = e^{\pm i\alpha + i\phi_{kl}}\tilde{\tau}_{kl}\tilde{\tau}^{\pm}_{k} \qquad \tilde{\tau}^{\pm}_{k}\tilde{\tau}_{lk} = e^{\mp i\alpha - i\phi_{kl}}\tilde{\tau}_{lk}\tilde{\tau}^{\pm}_{k} \tag{4.3}$$

$$\tilde{\tau}_{kl}\tilde{\tau}_{km} = e^{i\epsilon\alpha + i\phi_{lm}}\tilde{\tau}_{km}\tilde{\tau}_{kl} \qquad \tilde{\tau}_{kl}\tilde{\tau}_{ml} = e^{-i\epsilon\alpha - i\phi_{km}}\tilde{\tau}_{ml}\tilde{\tau}_{kl} \tag{4.4}$$

while the relations involving commutators are distorted as

$$e^{i\phi_{mk}}\tilde{\tau}^{\pm}_{k}\tilde{\tau}_{lm} - e^{i\phi_{lk}}\tilde{\tau}_{lm}\tilde{\tau}^{\pm}_{k} = 0 \tag{4.5a}$$

$$e^{i\phi_{kl}}\tilde{\tau}_{kl}\tilde{\tau}_{lk} - e^{-i\phi_{kl}}\tilde{\tau}_{lk}\tilde{\tau}_{kl} = 2i\sin\alpha(\tilde{\tau}^{-}_{k}\tilde{\tau}^{+}_{l} - \tilde{\tau}^{+}_{k}\tilde{\tau}^{-}_{l}) \tag{4.5b}$$

$$e^{i\phi_{mk}}\tilde{\tau}_{mk}\tilde{\tau}_{kl} - e^{i\phi_{kl}}\tilde{\tau}_{kl}\tilde{\tau}_{mk} = 2i\epsilon\sin\alpha\,\tilde{\tau}^{(\epsilon)}_{k}\tilde{\tau}_{ml} \tag{4.6}$$

$$e^{i\phi_{km}}\tilde{\tau}_{kl}\tilde{\tau}_{mn} - e^{i\phi_{ln}}\tilde{\tau}_{mn}\tilde{\tau}_{kl} = 2i\rho\sin\alpha\,\tilde{\tau}_{ml}\tilde{\tau}_{kn}\,. \tag{4.7}$$

Remarkably we may find again a realization for this multiparameter deformed case in q-oscillators, just from (3.9) through the following transformation

$$\begin{aligned}\tilde{\tau}^{\pm}_{k} &= \exp\Big[i\sum_{j\neq k}\phi_{kj}n_{j}\Big]\tau^{\pm}_{k}\\ \tilde{\tau}_{kl} &= \exp\Big[\frac{i}{2}\Big(-\phi_{kl} + \sum_{j\neq k}\phi_{kj}n_{j} + \sum_{j\neq l}\phi_{lj}n_{j}\Big)\Big]\tau_{kl}\end{aligned} \tag{4.8}$$

which consequently gives another realization of the FRT algebra through q-oscillators including deforming parameters. We also like to note that realization (4.8), through the connection between q-oscillators and q-groups [15, 20], provides the possibility of realising the FRT algebra in quantum group generators of $U_q(SU(N))$ in the multiparametric case. One observes further that the realization (4.8) when inserted in (3.4), can generate quantum-integrable models involving q-oscillators having the Lax operator (3.1*b*) corresponding to the multiparametrized R-matrix (4.2). Moreover, it is also evident that the deformed algebra (4.3)–(4.7) gets much simplified for the particular choice of parameters $\phi_{kl} = \pm\alpha$ and interestingly, this particular situation was found to be significant in the context of integrable models like Ablowitz–Ladik model, relativistic Toda chain etc [12].

5. Concluding remarks

In constrast to the usual Yang–Baxterization of the Braid group representation leading to YBE (1.4), we have discussed here a Yang–Baxterization scheme of the Faddeev–Reshetikhin–Takhtajan algebra leading to quantum YBE (1.3), a central relation in quantum-integrable systems. This scheme is considered for a class of R-matrices satisfying the Hecke relations and applied in particular to the $gl(N)$ case along with its multiparametric deformations. As a by-product a spectral parameter-dependent multiparametrized R-matrix is constructed. An intimate connection of FRT realtions with the extended trigonometric Sklyanin algebra and its multiparameter deformation is revealed, which helps us to find interesting realizations of the FRT algebra through q-oscillators. Using such realizations and with the help of the Yang–Baxterization procedure discussed here, one may construct Lax operators potentially important in the context of quantum integrable systems [12, 21].

Recently a non-standard braid group representation has been considered resulting in a new formulation of quantum group through FRT algebra [22]. However, since such exotic types of R-matrices [22, 23] obey the extra Hecke condition, the present approach is also applicable to them and might lead to new quantum-integrable models. It should also be interesting to investigate possible extensions of this scheme for the braid group representations satisfying other relations like Birman–Murakami–Wenzl algebra [13].

References

[1] de Vega H J 1989 *Int. J. Mod. Phys.* A **4** 2371
Witten E 1990 *Nucl. Phys.* B **330** 285
Alvarez-Gaume L, Gomes C and Sierra G 1989 *Nucl. Phys.* B **319** 155
Aref'eva I Ya and Volovich I V 1991 *Mod. Phys. Lett.* A **6** 893
[2] Kulish P P and Reshetikhin N Yu 1983 *J. Sov. Math.* **23** 2435
[3] Sklyanin E K 1982 *Funct. Anal. Appl.* **16** 27
[4] Jimbo M 1985 *Lett. Math. Phys.* **10** 63
[5] Drinfeld V G 1986 *Quantum Groups (Proc. ICM (Berkeley))* p 798
[6] Faddeev L, Reshetikhin N and Takhtajan L 1989 *Braid Groups, Knot Theory and Statistical Mechanics* ed C N Yang and M L Lie (Singapore: World Scientific) p 97
Faddeev L D 1990 *Fields and Particles* ed H Mitter *et al* (Berlin: Springer) p 89
[7] Reshetikhin N Yu, Takhtajan L A and Faddeev L D 1989 *Algebra and Analysis* vol I p 178
[8] Burroughs N 1990 *Commun. Math. Phys.* **133** 91
[9] Demidov D D, Manin Yu I, Mukhin E E and Zhdanovich D V 1990 *Preprint* Kyoto University RIMS-70
Sudbey A 1990 *J. Phys. A: Math. Gen.* **23** L697
Schirrmacher A, Wess J and Zumino B 1991 *Z. Phys.* C **49** 317
Fairlie D B and Zachos C K 1991 *Phys. Lett.* **256B** 43
[10] Reshetikhin N 1990 *Lett. Math. Phys.* **20** 331
[11] Schirrmacher A 1991 Multiparameter R-matrices and its quantum groups *Preprint* Max Plank Inst. MPI-ph/91-24
[12] Kundu A and Basu Mallick B 1991 Construction of integrable quantum lattice models through Sklyanin-like algebras *Mod. Phys. Lett.* A at press
Kundu A and Basu Mallick B 1991 Quantum groups in action: generation of quantum integrable models *Preprint* Saha Inst. SINP/TNP/91-23
[13] Jones V F R 1990 *Int. J. Mod. Phys.* B **4** 701
[14] Ge M L, Wang L Y and Kong X P 1991 *J. Phys. A: Math. Gen.* **24** 569
Cheng Y, Ge M L and Xue K 1991 *Commun. Math. Phys.* **136** 195
[15] Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581

Biederharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873
Sun C P and Fu H C 1989 *J. Phys. A: Math. Gen.* **22** L983
Hayashi T 1990 *Commun. Math. Phys.* **127** 129
[16] Sogo K, Akutsu Y and Abe T 1983 *Prog. Theor. Phys.* **70** 730
[17] Bernard D and Leclair A 1990 *Nucl. Phys.* B **340** 721
Saleur H and Zuber J B 1990 *Preprint* Saclay SPhT/90-071
[18] Babelon O 1984 *Nucl. Phys.* B **230** 241
[19] Chakrabarty R and Jagannathan R 1991 *J. Phys. A: Math. Gen.* **24** 1709
[20] Basu Mallick B and Kundu A 1991 *Mod. Phys. Lett.* A **6** 701
Kundu A and Basu Mallick B 1991 *Phys. Lett.* **156A** 175
[21] Basu Mallick B and Kundu A 1990 Role of quantum group in generating integrable field models *Preprint* Saha Inst. SINP/TNP/90-23
[22] Jing N and Ge M L 1991 *Lett. Math. Phys.* **21** 193
[23] Hlavaty L 1987 *J. Phys. A: Math. Gen.* **20** 1661